

# Theory of energy evolution in laser resonators with saturated gain and non-saturated loss

S. K. Turitsyn

Photonics Research Group, Department of Electronic Engineering, Aston University, Birmingham B4 7ET, UK.  
s.k.turitsyn@aston.ac.uk

**Abstract:** Theory of the energy evolution in laser resonators with saturated gain and non-saturated loss is revisited. An explicit analytical expression for the output energy/average power in terms of the gain saturation energy, cavity loss and small signal gain parameters is derived for a ring cavity configuration.

©2009 Optical Society of America

OCIS codes: (190.4360) Nonlinear optics, devices; (140.3430) Laser theory

---

## References and Links

1. A. E. Siegman, *Lasers* (University Science Books, 1986).
2. L. F. Mollenauer, and J. P. Gordon, *Solitons in Optical Fibers: Fundamentals and Applications* (Academic Press (2006)).
3. A. C. Newell, and J. V. Moloney, *Nonlinear Optics* (Addison-Wesley Publishing Company, Redwood City, CA, 1992).
4. A. Hasegawa, and Y. Kodama, *Solitons in Optical Communications* (Clarendon, Oxford, 1995).
5. V. E. Zakharov, and E. S. Wabnitz, *Optical Solitons: Theoretical Challenges and Industrial Perspectives* (Springer-Verlag, Heidelberg, 1998).
6. H. A. Haus, "Theory of mode locking with a slow saturable absorber," *IEEE J. Quantum Electron.* **11**(9), 736–746 (1975).
7. H. A. Haus, J. G. Fujimoto, and E. P. Ippen, "Analytic theory of additive pulse and Kerr lens mode locking," *IEEE J. Quantum Electron.* **28**(10), 2086–2096 (1992), Sh. Namiki, E. P. Ippen, H. A. Haus, and C. X. Yu, "C. X. Yu, "Energy rate equations for mode-locked lasers," *J. Opt. Soc. Am. B* **14**(8), 2099 (1997).
8. A. M. Dunlop, W. J. Firth, D. R. Heatley, and E. M. Wright, "Generalized mean-field or master equation for nonlinear cavities with transverse effects," *Opt. Lett.* **21**(11), 770–772 (1996).
9. J. N. Kutz, "Mode-locked soliton lasers," *SIAM Rev.* **48**(4), 629–678 (2006).
10. N. Akhmediev, and A. Ankiewicz, eds., *Dissipative Solitons, Lecture Notes in Physics*, (Springer, Berlin-Heidelberg, 2005) Vol. 661.
11. N. Akhmediev, and A. Ankiewicz, eds., *Dissipative Solitons: From optics to biology and medicine*, Lecture Notes in Physics, (Springer, Berlin-Heidelberg, 2008). Vol. 751.
12. J. M. Soto-Crespo, N. N. Akhmediev, V. V. Afanasjev, and S. Wabnitz, "Pulse solutions of the cubic-quintic complex Ginzburg-Landau equation in the case of normal dispersion," *Phys. Rev. E Stat. Phys. Plasmas Fluids Relat. Interdiscip. Topics* **55**(4), 4783–4796 (1997).
13. A. I. Chernykh, and S. K. Turitsyn, "Soliton and collapse regimes of pulse generation in passively mode-locking laser systems," *Opt. Lett.* **20**(4), 398 (1995).
14. L. Kramer, E. A. Kuznetsov, S. Popp, and S. K. Turitsyn, "Optical pulse collapse in defocusing active medium," *Pisma ZETF* **61**, 887–892 (1995) ("JETP Lett. **61**, 904 (1995)).
15. I. S. Aranson, and L. Kramer, "The world of the complex Ginzburg Landau equation," *Rev. Mod. Phys.* **74**(1), 99–143 (2002).
16. N. N. Rosanov, *Spatial Hysteresis and Optical Patterns* (Springer, Berlin, 2002).
17. W. W. Rigrod, "Saturation effects in high-gain lasers," *J. Appl. Phys.* **36**(8), 2487–2490 (1965).
18. A. Chong, W. H. Renninger, and F. W. Wise, "All-normal-dispersion femtosecond fiber laser with pulse energy above 20nJ," *Opt. Lett.* **32**, 2408–2410 (2007)

---

## 1. Introduction

There exist a wide variety of laser systems that cater to a range of applications in very diverse areas of science and industry. However, despite multiplicity of configurations, material bases and applications, the underlying operational and design principles appear to be the same from one device to another, and, thus, similar basic models can be applied to rather different families of lasers [1]. Performance and operation of many modern high power/energy advanced lasers are determined by a rather complex interplay between a number of physical effects that include gain, loss, dispersion and nonlinearity. Effects of saturation, the spectral

dependence of gain and loss, and the nonlinear dynamics of radiation in the laser cavity make such lasers even more complex physical systems. Therefore, design optimization and modeling of such lasers presents a multi-parametric nonlinear problem and any analytical or semi-analytical results are very important to reduce the number of optimization parameters and calibrate the model coefficients for comparison with experimental data.

Spatial evolution of the average power in CW lasers or pulse energy in pulsed lasers along the cavity is one of the important characteristics of system operation. Spatial dynamics of the energy/average power in a typical laser system is defined by the effects of gain provided by an amplifying medium; cavity losses that include both distributed and point losses, and effects of saturation of gain and loss. Note that in general, gain and loss can be frequency dependent either due to the spectral dependence of the gain/loss or as a result of optical filtering. Of particular interest are mathematical models describing high power mode-locked laser systems that present examples of nonlinear dissipative systems in which stable structures (dissipative solitons) emerge as a result of the balance between effects of gain/loss, dispersion and nonlinearity. Evidently, due to complex nature of such systems, there is a variety of possible mechanisms of formation of stable pulses. Note that typically, in many practical systems, the effect of the spectral dependence of the gain is small compared to other effects and does not play a critical role in the generation of stable pulses. The effect of the saturable absorber, on the contrary, is important for self-starting of the mode-locking. However, despite being of critical importance for generation of coherent structures from noise, in some systems and operational regimes, the saturated loss is not effectively involved in stabilization of pulse parameters at high energies. In this work we consider an energy balance equation in systems with dispersion, instantaneous nonlinearity (e.g. Kerr-type nonlinearity), saturated gain and non-saturated loss. The derived analytical results provide useful insight into the evolution of the energy along the cavity. However, we would like to stress the limitations of the considered, relatively simple mathematical model. Obviously, in practical laser schemes, especially at high-powers, additional physical constraints such as e.g. thermal effects, optical damage limits, pumping source brightness, mode overlap and others have to be taken into account. Note also, that our results are directly relevant to the mode-locking and/or Q-switched fibre laser systems, but the derived theory is rather general and can be applied in a variety of laser applications.

We consider here the master model that is often called either the nonlinear Schrödinger equation with dissipative terms [2–5] or the Ginzburg-Landau equations [6–16]. In the context of laser applications this model describes the temporal and longitudinal evolution of the slowly varying optical field envelope  $A(z,t)$  along the cavity medium (e.g. the active/passive fiber segments) in which the effects of point laser system elements (e.g. splicing loss) might be included as the corresponding distributed effects:

$$i \frac{\partial A}{\partial z} + \sum_{n=2}^{\infty} \frac{(i)^n \beta_n}{n!} \frac{\partial^n A}{\partial t^n} + \gamma |A|^2 A = i \left( \frac{g(A)}{2} - \frac{\alpha}{2} \right) A \quad (1)$$

Here the dispersive terms include higher-order dispersion expressed through derivatives of the propagation constant  $\beta(\omega)$  with respect of omega,  $\gamma$  is the nonlinearity parameter,  $g$  is the power dependent (saturated) gain and  $\alpha$  is an effective distributed loss. The gain is saturated in the following manner

$$g(A) = \frac{g_0}{1 + P_{av} / P_{sat}} = \frac{g_0}{1 + E / E_{sat}},$$

$$P_{av} = \frac{1}{T_R} \int |A(t)|^2 dt, \quad E = P_{av} T_R, \quad E_{sat} = P_{sat} T_R.$$

Here  $TR = 2nL/c$  is a round trip time, a homogeneous broadening is assumed,  $g_0$  is a small signal gain,  $P_{sat}$  is the saturation power (that in the context of fibre lasers is a product of the saturation intensity by the effective mode area) – of the signal power passing through the amplifying medium that saturates gain down to half of the small-signal gain  $g_0$ . Here saturation effects are intentionally presented either through the pulse energy or using the average power, to reflect the fact that the obtained results can be applied both in the context of CW lasers (when it is more appropriate to use average power) and in pulsed lasers. Note that the model (1) does not include terms responsible for the action of saturable absorber and for in-cavity filtering and in this sense is less general than the master equation introduced by Haus [6,7] or the model considered in [8] based on using the so-called ABCD approach [1]. In modern powerful laser systems the effects of nonlinearity and dispersion per cavity round trip might not be small resulting in a strong spectral and temporal evolution during pulse propagation through the laser resonator. Generation of laser radiation is determined by the complex interactions of different physical effects described by the master Eq. (1), with dispersion and Kerr nonlinearity playing critical roles. However, dispersive and nonlinear effects in Eq. (1) do not impact the energy of the radiation. In other terms, without the gain/loss dissipative term Eq. (1) is of the Hamiltonian type and the total energy is conserved. Evolution of the pulse energy (average power) along the amplifying medium with an effective distributed loss is governed by the ordinary differential equation that can be applied to both CW and pulsed lasers:

$$\frac{dE}{dz} = \frac{g_0 E}{1 + E/E_{sat}} - \alpha E \quad (2)$$

Note that this equation has been well studied in context of CW lasers and the general solution has been already described in the literature [1] and often is called a Rigrod analysis [17]. In this work we revisit these results in the context of specific cavity configurations to derive an analytical expression linking the output energy/average power and the cavity gain/loss parameters. Note that such exact energy evolution analysis, in general, cannot be applied in models used in [6–8]. Once more, it is important to point out that in many laser systems, the saturable absorber is critically important for initiation of generation, but does not have strong impact on the shaping of the generated pulses and the main features of the energy evolution along the cavity. We would like to point out that the obtained results are not directly applicable in systems where strong filtering is playing important role in shaping the generated laser pulse, for instance as in [18]. Note also, that our results are mostly applied in the situations when generated radiation propagates in one direction (e.g. ring cavity), however, the derived results can be also used under certain assumptions in more general configurations. Without loss of generality, we will refer in what follows to the pulse energy, having in mind that similar results can be derived for the averaged power in CW laser systems. First of all, it

is seen that the asymptotic stationary value of the energy is  $E_{asym} = E_{sat} \times [\frac{g_0 - \alpha}{\alpha}]$ . Integration of this equation yields the transcendent equation expressing energy as a function of the propagation distance  $z$  along the cavity:

$$\frac{E(z)}{E_{sat}} \left[ 1 - s \left( 1 + \frac{E(z)}{E_{sat}} \right) \right]^{-\frac{1}{s}} = \exp[(g_0 - \alpha)(z - z_0)], \quad s = \frac{\alpha}{g_0}. \quad (3)$$

Here  $z_0$  is a conserved quantity (integral of motion) of Eq. (1) that is determined by the initial value of the energy at  $z = 0$ . Evidently, in the limit  $\alpha \rightarrow 0$  Eq. (3) diverges into the known

relation:  $\ln \frac{E}{E_{sat}} + 1 + \frac{E}{E_{sat}} = g_0(z - z_0)$  [1]. The Eq. (3) allows us to express energy at any point along the cavity through the accumulated gain and the initial energy:

$$\frac{E(z)}{E_{sat}} \left[ 1 - s \left( 1 + \frac{E(z)}{E_{sat}} \right) \right]^{-\frac{1}{s}} = \exp[(g_0 - \alpha)(z - z_0)] = G(z) \times \frac{E(0)}{E_{sat}} \left[ 1 - s \left( 1 + \frac{E(0)}{E_{sat}} \right) \right]^{-\frac{1}{s}}.$$

It is convenient to introduce a function of two variables  $f_s(x) = x \times [1 - s - sx]^{-\frac{1}{s}}$  with  $s$  and  $x$  varying within the intervals  $0 < x < \frac{1-s}{s}$ ,  $0 < s < 1$ . The function  $f_s(x)$  has an

evident limit  $s \rightarrow 0 \Rightarrow f_s(x) \rightarrow x \exp(+x)$ . The inverse function  $f_s^{-1}(x)$  can be easily tabulated and used in numerical simulations in a way similar to any elementary functions. Figure 1 shows typical behaviour of the inverse function  $f_s^{-1}(y)$  with growing argument  $y$  for different values of the parameter  $s$ . It is important to note that the inverse function is calculated and tabulated once and forever and should not be recalculated each time it appears in equations.

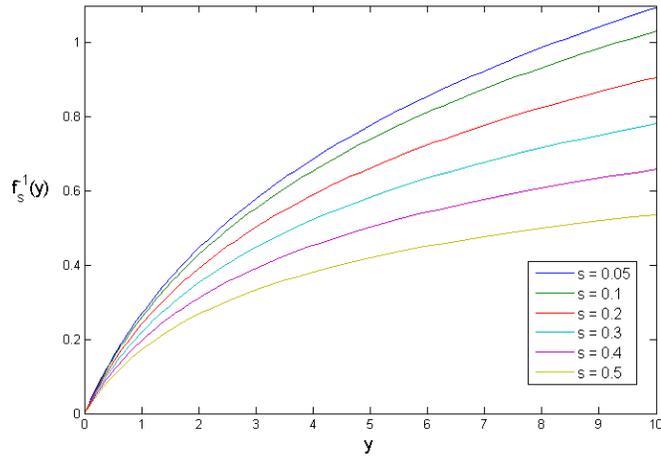


Fig. 1. The inverse function  $x = f_s^{-1}(y)$  for different values of the parameter  $s$ .

Evolution of the energy along the cavity then can be presented in a compact form:

$$E(z) = E_{sat} \times f_s^{-1} \left[ G(z) f_s \left( \frac{E(0)}{E_{sat}} \right) \right].$$

This expression can be used to present the balance of gain/loss in different configurations of the laser cavity. It is particular useful for the analysis of the energy mapping problem – evolution of the energy after one round trip and radiation shedding at the output coupler. As an example, we consider a Fabry-Perot cavity scheme where the action of the reflectors at the edges of the cavity is described by the transformation  $E^-(L) = R_1 E^+(L)$  and  $E^+(0) = R_2 E^-(0)$ .

Introducing the function  $f_s(x) = x \times [1 - s - sx]^{-\frac{1}{s}}$  allows us to express energy mapping after one round trip  $\{E^+(0) \rightarrow E^+(L) \rightarrow E^-(L) \rightarrow E^-(0) \rightarrow E^+(0)\}$  that includes the effects of the two

mirrors (e.g. fibre Bragg gratings in the case of fibre lasers) with the reflectivities  $R_1, R_2$  in a compact form:

$$\frac{E(0)}{E_{sat}} = R_2 \times f_s^{-1} \{ G f_s [ R_1 f_s^{-1} ( G f_s (\frac{E(0)}{E_{sat}}) ) ] \} \quad (4)$$

Here  $G = G(L)$  is one pass cavity gain. The transcendent Eq. (4) is not just a mathematical trick allowing to present energy evolution in the apparently closed way. This presentation greatly simplifies calculations of the energy dependence on  $G, R_1, R_2, E_{sat}, s = \alpha / g_0$  as a solution of the mapping problem (4). In the limit  $E / E_{sat} \ll 1$  the mapping Eq. (4) can be resolved analytically leading to the well-known laser resonator formula  $R_1 \times R_2 \times \exp[2(g_0 - \alpha)L] = 1$ .

For a ring laser configuration  $E_{out} = (1 - R)E(L)$  and  $E(0) = RE(L)$ , where  $L$  is a length of the ring cavity. In this case, one can derive an analytic expression linking laser output characteristic (energy or average power) with cavity and amplifying medium parameters:

$$E_{out} = E_{sat} \times \frac{g_0 - \alpha}{\alpha} \times \frac{1 - R}{\sqrt{R}} \times \frac{\sinh \left\{ \frac{\alpha [L(g_0 - \alpha) + \ln R]}{2g_0} \right\}}{\sinh \left\{ \frac{\alpha L(g_0 - \alpha) + (\alpha - g_0) \ln R}{2g_0} \right\}} \quad (5)$$

This compact analytical result connecting the key system and generated radiation characteristics can be used for calibration of parameters. Note that the analytical formula (5) is derived under assumption of unidirectional ring resonator and homogeneous broadening. In the limit  $\alpha \rightarrow 0$  the expression reproduces the well-known formula [1]:  $E_{out} = E_{sat} \times [\ln R + g_0 L]$ . Note that in the case of a long enough cavity  $L \rightarrow \infty$  (that can be practically defined as the resonator length larger than a scale at which the in-cavity energy reaches its asymptotic value the output energy is  $E_{asym} = E_{sat} \times [\frac{g_0 - \alpha}{\alpha}] \times (1 - R)$ .

Equation (5) can be re-written using an effective gain parameter,  $\Delta G(L) = L(g_0 - \alpha)$  assuming a net gain in the cavity:  $R \times \exp[\Delta G] \geq 1$  (this, for a given reflectivity restricts interval of allowed  $\Delta G$  to  $\exp[\Delta G] \geq 1/R$  and vice versa), and a loss/small gain ratio parameter  $s = \frac{\alpha}{g_0}$  in form:

$$E_{out} = (1 - R) \times E_{sat} \times \frac{1 - s}{s} \times \frac{1 - R^{-s} \times \exp[-s \Delta G(L)]}{1 - R^{1-s} \times \exp[-s \Delta G(L)]} = E_{out}(R, L, s) \quad (6)$$

Equation (5) is the key result of the work. This equation provides an analytic expression for the output energy/power as a function of laser system and amplifying medium parameters. It is seen that a large effective gain means that energy reaches an asymptotic value in one transit, mathematically this corresponds to the limit  $s \Delta G \gg 1$ , then the exponential terms can be neglected and the output energy is determined by the asymptotic value. Figures 2-4

illustrate Eq. (6) presenting the normalized energy  $E/E_{sat}$  as a function of  $s$ ,  $\Delta G$  ( $\Delta G [dB] = 10 \times \log_{10}(e^{\Delta G}) = 4.343 \times \Delta G$ ) and  $R$ .

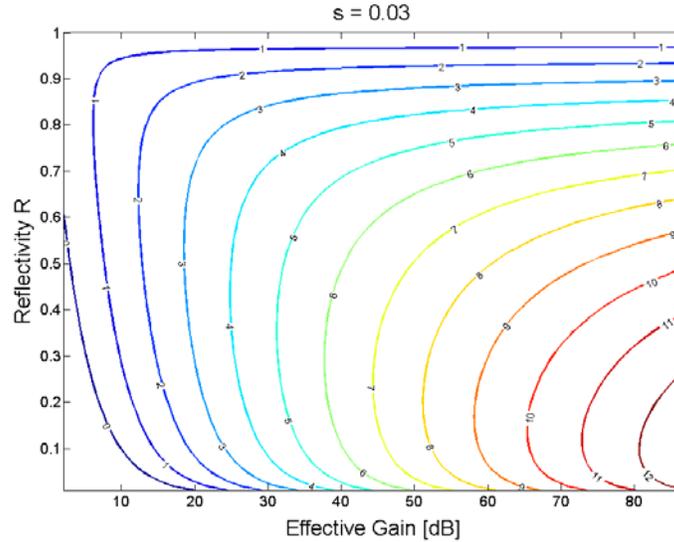


Fig. 2. Contour plot of the normalized energy  $E/E_{sat}$  in the plane (gain,  $R$ ); here  $s = 0.03$ .

Figure 2 presents dependence of the output energy on  $R$  and effective gain. Crosses of the horizontal lines (corresponding to fixed reflectivity) with the energy isolines (constant energy) show how much effective gain is needed to reach a certain output energy for a given  $R$ . For instance, it is seen that for reflectivity  $R$  in the range between 0.8 and 1, after certain value, further growth of the effective gain does not increase the output energy. The vertical tangent to the curve of constant output energy determines a minimal effective gain required to reach such output energy and a reflectivity  $R$  that is optimal (provides for lasing with minimal effective gain). Vertical lines corresponding to a fixed effective gain show that the same output energy can be achieved for two values of  $R$ .

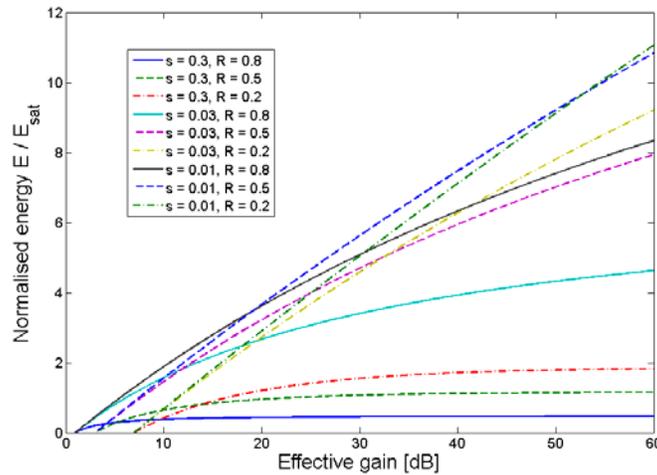


Fig. 3. Normalized energy  $E/E_{sat}$  as a function of an effective gain for different  $R$  and  $s$ .

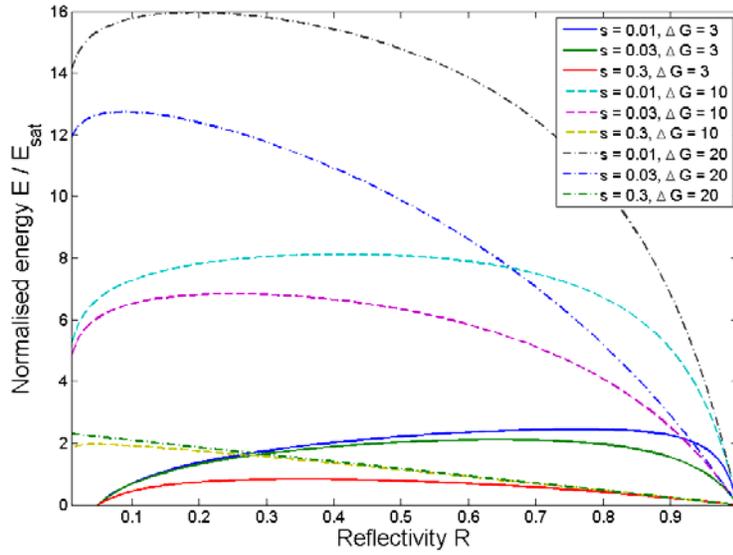


Fig. 4. Normalized energy  $E / E_{sat}$  vs. the reflectivity  $R$  for different gains and  $s$ .

Figure 3 illustrates that in the limit  $s(\Delta G + \ln R) < 1$ ,  $E_{out} / E_{sat} \rightarrow (1 - s) \times (\Delta G + \ln R)$ , (blue and green dashed lines); while in the opposite limit  $s\Delta G \gg 1$  (three first lines from the bottom):

$$E_{out} / E_{sat} \rightarrow (1 - R) \times \frac{1 - s}{s},$$

and this limit can be also seen in Fig. 4 (yellow dashed and green dot-dashed lines). An important result from Fig. 4 is that there are optimal out-coupling reflectivities  $R$  that provide the maximum output energy for a given effective gain. It is seen from Fig. 4 also that performance is strongly affected by the loss/gain parameter  $s = \alpha / g_0$ . In particular, the optimal reflectivity  $R$  can be shifted from values close to 1 as on solid blue line to very small  $R$  as for dashed green and yellow. Therefore, the ratio of non-saturated cavity loss to the small gain (saturated at larger energies) is an important physical parameter that should be taken into account in the design of such lasers.

Finally, note that a similar analytical expression can be derived for pulsed lasers based on the Fabry-Perot resonator with one highly reflective mirror ( $R_1 = 1$ ). The approximate expression for the energy at the location of the mirror for Fabry-Perot cavity is then the same as Eq. (5) with the following replacements:  $L \rightarrow 2L$ ,  $R \rightarrow R_2$ .

In conclusion, the energy mapping in laser resonators with saturated gain and non-saturated loss is examined. An analytical expression for the output energy/average power in terms of the gain saturation energy, cavity loss and small signal gain parameters is derived for a ring cavity configuration. The described approach is applicable to a wide range of CW and pulsed laser systems and can be used for reduction of the space of optimization parameters and/or for calibration of the model parameters against the measured system characteristics.

I would like to thank A. Latkin and A. Siegman for useful discussions and to acknowledge support of the Royal Society and EPSRC.