Dispersion compensation is a widely applied technique that is employed in long-distance transmission at high data rates to remove the effects of dispersion. For purely linear systems it is of no consequence where or how often within the transmission line the compensation is applied (at the start, at the end, or at an intermediate point in the system). For systems in which there is any degree of nonlinearity that is not the case, and the position and frequency of the compensation are vital considerations. This is particularly so for soliton and return-to-zero (RZ) transmission formats but is also an important consideration for non-RZ systems. Here we aim to show from elementary symmetry arguments that stable evolution of the signal requires a chirp-free point at the midpoint of the dispersive sections. It is now well known from extensive numerical and appropriate analytical models of nonlinear propagation that ideal periodic pulses can be obtained for a range of strengths of two-stage dispersion maps (see, e.g., Refs. 1–13 and references therein). These solutions are conveniently labeled solitons, as they retain many familiar soliton properties and can be traced back to the conventional solitons in the limit of weak dispersion management. In all cases the observation is that in a consideration of a periodic dispersive structure with two sections of opposite sign of dispersion (with arbitrarily different physical lengths) the solution requires that the pulses be unchirped at the midpoint of both sections. This is a profoundly important observation. It shows that the naive application of dispersion compensation with initially unchirped source pulses with exact dispersion compensation either periodically or at the end of the system is as far removed from the ideal solution as it is possible to get. The concept of prechirping the source pulses to match the stable chirp needed at the start of the transmission fiber is of course one clear way to deal with this problem. It is worth reemphasizing that if the systems were linear then the initial waveform would be immaterial because exact compensation guarantees restoration of whatever waveform is launched. For nonlinear systems such is not the case, and only a small nonlinearity is sufficient to destroy perfect reproducibility. An intuitive physical explanation of the observation that a chirp-free point (CFP) is at the center of each section in the two-step lossless map is as follows: About either CFP the evolution is symmetric, so pulse width and bandwidth are mirrored in the positive and negative z directions. The pulse width is minimum at the CFP for both anomalous and normal sections. However, the bandwidth is minimum in the normal and maximum in the anomalous section. Consider now moving the CFP from the center in any, say, the anomalous, section. If it is moved to larger z, the pulse width at the boundary is reduced and the bandwidth (at the boundary) increased. There is now inconsistency in where the CFP must be in the next section of normal dispersion. To return to the original pulse width at the end of the section the CFP must move to shorter z; however, to return to the same bandwidth it must move to longer z. This is inconsistent and thus shows that the CFP cannot be moved from the center by these simple symmetry arguments. The argument above applies to any pulse shape, and the only requirement is for periodicity. This physical argument applies only to two-stage maps, and below we give an example of a situation in which there is more than one solution consistent with the symmetry for a more complex map.

Next we present a more rigorous analysis of the symmetry arguments. We consider the fundamental lossless, dispersive nonlinear propagation model, \( \frac{\partial^2 A}{\partial z^2} + d(z) \frac{\partial^2 A}{\partial t^2} + \sigma |A|^2 A = 0 \), where \( d(z) \) is the dispersion (see Ref. 3 for notation and normalization), periodic with period \( L \). The dispersion-managed (DM) soliton solution to Eq. (1) is given (see, e.g., Ref. 3) by \( A(z, t) = F(z, t, k \exp(ikz)) \), with a periodic function \( F(z + L, t) = F(z, t) \). The soliton shape is given by

\[
\frac{\partial F}{\partial z} - kF + d(z) \frac{\partial^2 F}{\partial t^2} + \sigma |F|^2 F = 0.
\]  

In the most practical dispersion map, parameter \( k \) (wave number or quasi momentum in terms of the theory of the Bloch functions) completely and uniquely characterizes the DM soliton solution of Eq. (1). In practical terms, parameter \( k \) uniquely determines the pulse width and the energy of the DM soliton and is unambiguously deduced from the pulse width. We do
not consider here the degeneracy that corresponds to the two branches of the solutions that exist in the strong map with zero or normal average dispersion, which have different energies for the same pulse width. In general, for some specific maps it could happen that two stable periodic solutions do exist with the same pulse width (an example is presented below). However, for most practical maps we can conclude that it is already known (see, e.g., Refs. 3–5) that fitting (even if it is approximate) of the input signal with the true periodic waveform for a given map can be realized with a prechirping technique4 that uses an additional prechirping technique.

For completeness of presentation, we point out that the symmetry analysis developed above does not apply to the case of multistability; when two or more stable solutions exist with the same pulse width, then \( F^*(z, t, k) \neq F(z, t, k) \). There are dispersion maps for which a symmetry breaking occurs, in particular when the dispersion is small near the boundary. For instance, the specific dispersion map shown in Fig. 1 with segments of identical lengths and dispersions of \(-1, -2, -1, 1, 2\), and 1 has two solutions, which are complex conjugate at the beginning of the section. Both of these solutions are stable, as one can see from Fig. 1 (top), where slow stroboscopic evolution (at the ends of sections) of the pulse shapes is shown. The pulse width’s evolution over one period for two stable periodic solutions is also shown in Fig. 1 (bottom, solid curves). The dispersion map is plotted by dashed lines. Evolution from one to zero for the solution shown at the left \([F(z, t, k)]\) is the same as the dynamics of the solution at the right \([F^*(z, t, k)]\) from one to zero, in accordance with the symmetry of Eq. (1). One can see from Fig. 1 that, in this case, CFP's are not in the middles of fibers, even in this lossless case. This example demonstrates that such a mathematical property as uniqueness of the periodic solution could be important for the dynamics of a DM system.

The existence of the CFP's is a factor of crucial importance for design of DM transmission systems. It is already known (see, e.g., Refs. 3–5) that fitting (even if it is approximate) of the input signal with the true periodic waveform for a given map can be realized with a prechirping technique4 that uses an additional prechirping technique.

![Fig. 1](image)

**Fig. 1.** Two stable periodic solutions do exist in the dispersion map shown by dashed lines. Bottom, pulse width evolution (solid lines) over one period. Top, three-dimensional stroboscopic evolution (at the ends of the section), demonstrating the stability of both solutions. In this case CFP's that correspond to the minima of the pulse width are not in the centers of the symmetry. Distance is normalized by \( L \).
fiber, with phase modulation, or by launching of the transform-limited pulse at the specific CFP’s. The appropriate input chirp can be calculated from two basic ordinary differential equations (ODE’s) for the RZ signal width and chirp (see, e.g., Refs. 3, and 6, and 10). It is interesting to address the following question: What is the optimal point along the map from which to launch an input signal with parameters found by use of this approximate model? The answer to this question will determine whether there is any difference between launching a transform-limited signal at a CFP or an appropriately chirped pulse at any other point along the dispersion map. In other words, this is a question about sensitivity of the signal evolution to small deviations of input pulse from the true periodic waveform. This issue is of evident practical interest because in the practical system it is hard to form input signals that have perfect DM soliton shapes. Recall that a conventional soliton can be formed from a wide range of initial RZ signals. This feature has not been well studied for DM solitons, though we can expect that a RZ signal (for instance, a Gaussian pulse) with parameters close to those of a DM soliton will propagate stably or quasi stably over long distances.

Figure 2 shows slow (stroboscopic) evolution of the pulse peak power (top left) and the rms pulse width (bottom left) of Gaussian pulses with parameters found from the ODE model, launched either in the chirp-free point (solid curves) or at the junction between two fibers (dashed curves). In the second case the pulse has the appropriate chirp. Here the normalized $d(z)$ is 5.15 in the first fiber and 4.85 in the second fiber, $a = 1$, and other normalized parameters are shown in the figure. Figure 2, right, shows the slow (stroboscopic) dynamics as a mapping in the plane rms pulse width with rms chirp. The input signals launched at the CFP (solid curve) and at the junction between fibers (dashed curve) after some transition period evolve to quasi-stable states shown by the circles in Fig. 2 (right). The DM soliton presents a fixed point in this figure. It can be seen that the ODE approximation of the input signal works better when the pulse is launched at the CFP’s. This result has the following qualitative explanation: Near CFP’s the changes in the signal parameters are slower than the changes near the junction between two fibers; therefore it is natural to expect that the same small initial deviations from the periodic solution will lead to larger deflections in the corresponding trajectories, as shown in Fig. 2. This result, in particular, indicates that it might be better to use a proper prechirping fiber (starting from the CFP) rather than phase modulation to chirp the input RZ signal.

In conclusion, we have shown from an elementary symmetry analysis that in dispersion-compensated systems for which a lossless model is valid (for instance, when compensation period $L$ is much larger than amplification distance $Z_a$), nonlinearity requires a chirp-free point at the center of each section. We have also demonstrated that when one is using a Gaussian approximation for a dispersion-managed soliton it is advantageous to launch the signal at the chirp-free points of the map rather than to launch a chirped input signal at the beginning of the map.

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